

Smale Strategies for Network Prisoner's Dilemma Games

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Abstract

Smale's approach [13] to the classical two-players repeated Prisoner's Dilemma game is revisited here for N -players and Network games in the framework of Blackwell's approachability, stochastic approximations and differential inclusions.

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1 Introduction

It has been well known for many years that mutual cooperation is a Nash equilibrium outcome in a two players infinitely repeated Prisoner's Dilemma game, even though defection is the dominant strategy of the one-shot game (see e.g. the classical book by Axelrod [2]).

In 1980 Smale [13] studied the two players repeated Prisoner's Dilemma game under the assumption that both players have limited memory and only keep track of the cumulative average payoffs. In this setting, he showed that a very simple deterministic strategy called a *good strategy*, if adopted by one player, leads to cooperation, in the sense that the other player has interest to cooperate. A good strategy, as defined by Smale, is a strategy such that the player cooperates unless her average payoff to date is significantly less than her opponent. Later, Benaïm and Hirsch [6] considered the stochastic analogue of Smale's solution. In 2005, Benaïm Hofbauer and Sorin [7] using tools from stochastic approximation and differential inclusions showed that the results of Smale, Benaïm and Hirsch can be reinterpreted in the framework of Blackwell's approachability theory [9], and that the assumption that "both" players keep track only of the cumulative average payoff is unnecessary.

The present paper extends these works to variant of the classical Prisoner Dilemma game including N -Players where the underlying structure is a network. It is based on K. Abhyankar's PhD thesis [1], Blackwell's approachability [9] and the stochastic approximation approach to differential inclusions developed in [7].

Section 2 sets up the notation and reviews briefly Blackwell's approachability and some of the results in [7]. Section 3 considers N -players prisoner dilemma games and Section 4 prisoner dilemma games in which players are located at the vertices of a symmetric graph and interact only with their neighbors. Smale good strategies are defined for these games and are shown to be Nash equilibria.

2 Notation and Background

Let A and B be two finite sets representing respectively the *action sets* of some *decision maker DM* (for instance a player, or a group of players) and the action set of *Nature* (for instance the player's opponents). Let $U : A \times B \mapsto \mathbb{R}^N$ be a vector valued *payoff function*.

Throughout, we let $E \subset \mathbb{R}^N$ denote the convex hull of the payoff vectors

$$E = \text{conv}\{U(a, b) : a \in A, b \in B\}.$$

At discrete times $n = 1, 2, \dots$, DM and Nature choose their actions $(a_n, b_n) \in A \times B$. We assume that:

- (a) The sequence $\{(a_n, b_n)\}_{n \geq 0}$ is a random process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and adapted to some filtration $\{\mathcal{F}_n\}$ (i.e. $\{\mathcal{F}_n\}$ is an increasing family of sub- σ fields of \mathcal{F} , and for each n (a_n, b_n) is \mathcal{F}_n measurable). Here \mathcal{F}_n has to be understood as the history up to time n .

- (b) Given the history \mathcal{F}_n , DM and Nature act independently:

$$\mathbf{P}((a_{n+1}, b_{n+1}) = (a, b) | \mathcal{F}_n) = \mathbf{P}(a_{n+1} = a | \mathcal{F}_n) \mathbf{P}(b_{n+1} = b | \mathcal{F}_n).$$

Let $\mathcal{P}(A)$ (respectively $\mathcal{P}(B)$) denote the set of probabilities over A (respectively, B).

A (long term) *strategy* for DM is a stochastic process $\Theta = \{\Theta_n\}$ adapted to $\{\mathcal{F}_n\}$ taking values in $\mathcal{P}(A)$. We say that DM uses strategy Θ if

$$\Theta_n(a) = \mathbf{P}(a_{n+1} = a | \mathcal{F}_n) \tag{1}$$

for all $a \in A$.

The *cumulative average payoff* at time n is the vector

$$u_n = \frac{1}{n} \sum_{k=1}^n U(a_k, b_k) \in E. \tag{2}$$

Strategy Θ is said to be *payoff-based* provided

$$\Theta_n(a) = Q_{u_n}(a)$$

for all $a \in A$, where for each $u \in E$, $Q_u(\cdot)$ is a probability over A and $u \in E \mapsto Q_u \in \mathcal{P}(A)$ is measurable. In this case, the family $Q = \{Q_u\}_{u \in E}$ is identified with DM 's *strategy*.

Example 1 (M -Players games) Consider an M -players game with $M \geq 2$. Players are denoted $i = 1, \dots, M$. Player i has a finite *action set* (or *pure*

strategy set) denoted Σ^i , and a payoff function $U^i : \Sigma^1 \times \dots \times \Sigma^M \mapsto \mathbb{R}^{N_i}$ for some $N_i \geq 1$.

At each discrete time $n = 1, 2, \dots$ Player i chooses an action $s_n^i \in \Sigma^i$ and receives the payoff $U^i(s_n^1, \dots, s_n^M)$.

Choose DM to be some given subset of players, say $I = \{1, \dots, k\}$. Set

$$A = \Sigma^1 \times \dots \times \Sigma^k, B = \Sigma^{k+1} \times \dots \times \Sigma^M$$

and $U : \Sigma \mapsto \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_k} \simeq \mathbb{R}^N$, $N = \sum_{i=1}^M N_i$, with

$$U(a, b) = (U^1(a, b), \dots, U^M(a, b)).$$

◇

The Limit Set theorem

Assume that DM has a payoff-based strategy Q . For each $u \in E$ let

$$C(u) = \left\{ \sum_{a \in A, b \in B} U(a, b) Q_u(a) \nu(b) : \nu \in \mathcal{P}(B) \right\}. \quad (3)$$

The set $C(u)$ is the convex set containing all the average payoffs that are obtained when DM plays the mixed strategy Q_u and Nature plays any mixed strategy.

Let $\mathbf{C} \subset E \times E$ be the intersection of all closed subset $\mathbf{G} \subset E \times E$ for which the fiber $\{y \in E : (x, y) \in \mathbf{G}\}$ is convex and contains $C(x)$. The *closed-convex* extension of C , denoted $\overline{\text{co}}(C)$ is defined as

$$\overline{\text{co}}(C)(x) = \{y \in E : (x, y) \in \mathbf{C}\}.$$

For convenience we extend $\overline{\text{co}}(C)$ to a set-valued map $\overline{\text{co}}(C)$ on \mathbb{R}^N , also denoted $\overline{\text{co}}(C)$, by setting

$$\overline{\text{co}}(C)(x) = \overline{\text{co}}(C)(r(x)). \quad (4)$$

where for all $x \in \mathbb{R}^N$, $r(x) \in E$ denotes the unique point in E closest to x . Associated to $\overline{\text{co}}(C)$ is the differential inclusion

$$\frac{du}{dt} \in F(u) := -u + \overline{\text{co}}(C)(u). \quad (5)$$

A *solution* to (5) is an absolutely continuous mapping $t \mapsto \eta(t)$ verifying $\dot{\eta}(t) \in F(\eta(t))$ for almost every $t \in \mathbb{R}$. Given such a solution, its *initial condition* is the point $\eta(0)$. Throughout, we let $S_u \subset C^0(\mathbb{R}, \mathbb{R}^N)$ denote the set of all solutions to (5) with initial condition u . By construction, F maps points to non empty compact convex sets and has a closed graph. Thus, by standard results on differential inclusions, S_u is a nonempty subset of $C^0(\mathbb{R}, \mathbb{R}^N)$ that is compact (for the topology of uniform convergence on compact intervals) and (5) induces a set-valued dynamical system $\Phi = \{\Phi_t\}$ defined for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^N$ by

$$\Phi_t(u) = \{\eta(t) : \eta \in S_u\}.$$

A set $\Lambda \subset \mathbb{R}^N$ is said to be *invariant* for (5) if for all $u \in \Lambda$ there exists $\eta \in S_u$ such that $\eta(\mathbb{R}) \subset \Lambda$ (see section 3 of [7] for other notions of invariance, more details and references on set valued dynamics).

A nonempty compact set Λ is called an *attracting set* for Φ provided there is some neighborhood U of Λ , called a *fundamental neighborhood*, with the property that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\Phi_t(U) \subset N^\varepsilon(\Lambda)$ for all $t \geq t_\varepsilon$. Here N^ε stands for the ε neighborhood of Λ . If in addition Λ is invariant, Λ is called an *attractor*. By Proposition 3.10 in [7], every attracting set contains an attractor with the same fundamental neighborhood.

The *basin of attraction* of an attracting set Λ is the set

$$\mathcal{W}(\Lambda) = \{u \in \mathbb{R}^N : \omega_\Phi(u) \subset \Lambda\}$$

where

$$\omega_\Phi(u) = \bigcap_{t \geq 0} \overline{\Phi_{[t, \infty[}(u)}.$$

We let

$$L = L(\{u_n\})$$

denote the limit set of the sequence $\{u_n\}$ defined by (2). Note that L is a random subset of E .

Point $p \in \mathbb{R}^N$ is called *attainable* if for any $n \in \mathbb{N}$ and any neighborhood U of p

$$\mathbf{P}(\exists m \geq n : u_m \in U) > 0.$$

We let $\text{Att}(\{u_n\})$ denote the set of attainable points.

Parts (i) and (ii) of the following result follow from Theorems 3.6 and 3.23 in [7], generalizing the limit set theorem obtained for stochastic approximation processes (associated to an ODE) in [3, 4] and asymptotic pseudotrajectories

(of an ODE) in [5]. Part (iii) follows from [10] generalizing a result obtained for stochastic approximation processes (associated to an ODE) in [4].

Theorem 2.1 *Suppose that DM uses the payoff-based strategy Q . Then with probability one (regardless of Nature strategy)*

- (i) $L = L(\{u_n\})$ is almost surely an internally chain-transitive set of (5).
- (ii) If Λ is an attracting set for Φ then $L \subset \Lambda$ on the event $L \cap \mathcal{W}(\Lambda) \neq \emptyset$,
- (iii) If $\text{Att}(\{u_n\}) \cap \mathcal{W}(\Lambda) \neq \emptyset$ then $\mathbf{P}(L \subset \Lambda) > 0$.

We refer the reader to [7] for the definition of "internally chain-transitive" sets since this notion will not be used here but for the fact that an internally chain-transitive set is compact and invariant under differential inclusions (5).

Approachability

Let d denote the Euclidean distance in \mathbb{R}^N . A set $\Lambda \subset E$ is said *approachable* if there exists a long term strategy for DM such that, regardless of Nature strategy,

$$d(u_n, \Lambda) \rightarrow 0.$$

Given a compact subset $\Lambda \subset E$ and $x \in E$, define

$$\Pi_\Lambda(x) = \{y \in \Lambda : d(x, \Lambda) = d(x, y)\}$$

where $d(x, \Lambda) = \inf\{d(x, y) : y \in \Lambda\}$.

Record that $N^r(\Lambda) = \{x \in E : d(x, \Lambda) < r\}$. We say that Λ is a *local \mathcal{B} -set* for the payoff-based strategy Q (or simply a local \mathcal{B} -set) if there exists $r > 0$ such that for all $x \in N^r(\Lambda) \setminus \Lambda$ there exists $y \in \Pi_\Lambda(x)$ such that the hyperplane orthogonal to $[x, y]$ at y separates x from $C(x)$. That is,

$$\langle x - y, v - y \rangle \leq 0 \tag{6}$$

for all $v \in C(x)$ as defined by (3). If Λ is a local \mathcal{B} -set for all $r > 0$ it is simply called a \mathcal{B} -set. Blackwell [9], proved that being a \mathcal{B} -set is a sufficient condition for approachability.

Theorem 2.2 *Let $\Lambda \subset E$ be a local \mathcal{B} -set for the payoff-based strategy Q . Then*

- (i) Λ contains an attractor for Φ with fundamental neighborhood $U = N^r(\Lambda)$.
In particular,
- (a) $L \subset \Lambda$ on the event $L \cap U \neq \emptyset$.
 - (b) $\text{Att}(\{u_n\}) \cap U \neq \emptyset \Rightarrow \mathbf{P}(\{L \subset \Lambda\}) > 0$.
- (ii) If Λ is a B-set, then $\mathbf{P}(L \subset \Lambda) = 1$.

Proof: It is proved in [7], Corollary 5.1 that Λ contains an attractor for (5) provided inequality (6) holds for all $v \in \overline{\text{co}}(C)(x)$ (rather than merely $v \in C(x)$). It then suffices to prove that (6) also holds for all $v \in \overline{\text{co}}(C)(x)$.

Let $\text{Graph}(C) = \{(x, y) \in E \times E : y \in C(x)\}$. Denote its closure by $\overline{\text{Graph}}(C)$, and set

$$\overline{\text{Graph}}_x(C) = \{y \in E : (x, y) \in \overline{\text{Graph}}(C)\}.$$

Let $D(x)$ be the convex hull of $\overline{\text{Graph}}_x(C)$. It follows from (6) and compactness of Λ that $\langle x - y, v - y \rangle \leq 0$ for all $x \in E, v \in \overline{\text{Graph}}_x(C)$ and some $y \in \Pi_\Lambda(x)$. Clearly, this inequality still holds for all $v \in D(x)$. We claim that $\overline{\text{co}}(C)(x) = D(x)$ from which the proof of (i) follows.

Proof of the claim: The inclusion $D(x) \subset \overline{\text{co}}(C)(x)$ follows from the definitions. To prove the opposite inclusion it suffices to verify that $\text{Graph}(D)$ is closed. Let $x_n \rightarrow x, y_n \rightarrow y$ with $y_n \in D(x_n)$.

By the Caratheodory Theorem (see e.g. Theorem 11.1.8.6 in [8]), the convex hull of a set $G \subset \mathbb{R}^N$ equates the set obtained by taking all convex combinations of $N + 1$ points in G . Thus, there exist

$$w_n = (w_{n,1}, \dots, w_{n,N+1}) \in \overline{\text{Graph}}_{x_n}(C)^{N+1}$$

and

$$\alpha_n = (\alpha_{n,1}, \dots, \alpha_{n,N+1}) \in \Delta^N \text{ (the unit } N\text{-dimensional simplex of } \mathbb{R}^{N+1}\text{)}$$

such that

$$y_n = \sum_{i=1}^{N+1} \alpha_{n,i} w_{n,i}.$$

By compactness, after replacing sequences by subsequences we can assume that $\alpha_n \rightarrow \alpha \in \Delta^N$ and $w_n \rightarrow w$. Closedness of $\overline{\text{Graph}}(C)$ ensures that $w \in \overline{\text{Graph}}_x(C)^{N+1}$. Thus $y \in D(x)$. This proves the claim.

Assertions (a) and (b) are now consequences of Theorem 2.1. The last statement was proved by Blackwell [9]. Note that it also follows from (i). \square

A straightforward application of this last theorem is given by the following result. It will be used several times in the forthcoming sections.

Let $\mu \in \mathbb{R}^N$ with $\mu \neq 0$. For all $x \in \mathbb{R}^N$, set $\mu(x) = \langle \mu, x \rangle$.

Corollary 2.3 *Suppose there exist actions $a_1, a_2 \in A$ and numbers α, β such that for all $b \in B$*

$$\mu(U(a_1, b)) \leq \alpha \text{ and } \mu(U(a_2, b)) \geq \beta.$$

Let Q be a payoff-based strategy such that

$$\mu(u) > \alpha \Rightarrow Q_u(a_1) = 1$$

and

$$\mu(u) < \beta \Rightarrow Q_u(a_2) = 1.$$

Then

$$\Lambda = \{u \in E : \mu(u) \in [\alpha, \beta]\}$$

is a B -set

Note that there is no assumption here that $\alpha \leq \beta$. If $\alpha \geq \beta$ $[\alpha, \beta]$ stands for $[\beta, \alpha]$.

Proof: Equation (6) in this context becomes

$$\mu(u) > \alpha \Rightarrow \mu(v) \leq \alpha$$

$$\mu(u) < \beta \Rightarrow \mu(v) \geq \beta,$$

for all $u \in E, v \in C(u)$. By convexity of the half spaces $\{\mu(v) \leq \alpha\}, \{\mu(v) \geq \beta\}$, and the definition of Q this is equivalent to the condition given in the statement of the corollary. \square

We conclude this section with some quantitative estimates given in the excellent recent survey paper by Perchet [12]. Let

$$|E| = \sup\{\|v\| : v \in E\}, \quad |\Lambda| = \sup\{\|v\| : v \in \Lambda\}.$$

The first assertion of the next theorem follows from Corollary 1.1 in [12]. It is slight variant of a result obtained by Blackwell [9]. The second assertion follows from Corollary 1.5 in [12].

Theorem 2.4 *Suppose DM adopts the payoff-based strategy Q and that $\Lambda \subset E$ is a \mathcal{B} -set for Q . Then for all $\eta > 0$*

(i)

$$\mathbb{P}(\sup_{m \geq n} d(u_m, \Lambda) \geq \eta) \leq \frac{2(|E| + |\Lambda|)^2}{\eta^2 n}$$

(ii) *If furthermore Λ is convex,*

$$\mathbb{P}(\sup_{m \geq n} d(u_m, \Lambda) - 2 \frac{|E|}{\sqrt{m}} \geq \eta) \leq 4 \exp\left(-\frac{\eta^2 n}{32|E|^2}\right).$$

3 N -Players Prisoner's Dilemma Game

Consider an N -Players game (as described in Example 1) where each player has two actions: *cooperate* C or *defect* D , so that $\Sigma^i = \{C, D\}$. We assume that the payoff functions $U^i, i = 1, \dots, N$, are as follows. Let $s = (s^1, \dots, s^N) \in \{C, D\}^N$ be the action profile of the players. If Player i cooperates (i.e $s^i = C$) and amongst her $N - 1$ opponents, k cooperate (i.e $\text{card}\{j \neq i : s^j = C\} = k$) she gets

$$U^i(s) = v(C, k).$$

If she defects and amongst her opponents, k cooperate she gets

$$U^i(s) = v(D, k).$$

For $k = 0, \dots, N - 1$ the numbers $v(C, k), v(D, k)$ satisfy the following conditions, usual for prisoner's dilemmas:

(i) Defection is the dominant action :

$$v(C, k) < v(D, k)$$

for all $k = 0, \dots, N - 1$;

(ii) The payoff of a defector increases with the number of cooperators :

$$v(D, k) \leq v(D, k + 1)$$

for all $k = 0, \dots, N - 2$;

(iii) Mutual cooperation is a Pareto optimal: For all $s \in \{C, D\}^N$

$$\sum_{i=1}^N U^i(s) \leq NU^i(C, \dots, C);$$

Or, equivalently, for all $k = 0, \dots, N - 1$

$$kv(C, k - 1) + (N - k)v(D, k) \leq Nv(C, N - 1).$$

(iv) (Occasional assumption) Mutual defection is Pareto inefficient: For all $k = 0, \dots, N - 1$

$$kv(C, k - 1) + (N - k)v(D, k) \geq Nv(D, 0).$$

Remark 1 Condition (i) makes (D, \dots, D) the unique Nash equilibrium of the one-shot game.

Remark 2 When $N = 2$ we retrieve the usual two players Prisoner's Dilemma game. The Pareto conditions (iii) and (iv) amount to say that the polygon with vertex set

$$\{(v(D, 0), v(D, 0)); (v(D, 1), v(C, 0)); (v(C, 0), v(D, 1)); (v(C, 1), v(C, 1))\}$$

is convex.

Example 2 (Free riding) Let $f : \{0, \dots, N\} \mapsto \mathbb{R}^+$ and $c > 0$ be such that

$$\frac{c}{N} \leq f(k + 1) - f(k) < c.$$

Let

$$v(C, k) = f(k + 1) - c \text{ and } v(D, k) = f(k).$$

This can be seen as a simple model of "free riding". Each player can either Contribute (Cooperate), or Defect from contributing, to a public good. Individual contribution costs c and everyone -even if a defector- benefits from the good and is paid $f(k)$, when there are k contributors.

Note that the assumption on f imply that conditions (i) – (iv) above are satisfied. The fact that mutual defection is a Nash equilibrium of the one-shot game is reminiscent of Hardin's book *The Tragedy of the Commons* [11].

◇

Recall that $E = \text{conv}\{U(s) : s \in \{C, D\}^N\}$. For $u = (u_1, \dots, u_N) \in E$ let

$$\mu^i(u) = u_i - \frac{1}{N-1} \sum_{j \neq i} u_j = \frac{1}{N-1} \sum_{j=1}^N (u_i - u_j).$$

Let δ be a nonnegative real number, Adapting [13], [6] and [7], we define a δ -good strategy for Player i as a payoff-based strategy Q^i (as defined in section 2) for the Decision Maker, Player i , such that

$$Q_u^i(C) = 1 \text{ if } \mu^i(u) \geq 0,$$

and

$$Q_u^i(D) = 1 \text{ if } \mu^i(u) < -\delta.$$

We call such a strategy *continuous* whenever the map $u \mapsto Q_u$ is continuous.

The following result shows that, by playing a δ -good strategy, a player (or a group of players) makes sure that her opponents' average payoff cannot be much better than hers, nor than the Pareto optimal payoff. Under the supplementary condition (iv) she ensures that her payoff cannot be much worse than the payoff resulting from mutual defection. If furthermore, all the players play a δ -good strategy, one of them being continuous, the outcome is the one given by mutual cooperation. As a consequence (Corollary 3.2), continuous δ -good strategies form a Nash equilibrium. The proof is postponed to the end of the section.

Theorem 3.1 *Let $k \leq N$. Suppose that for all $i \in \{1, \dots, k\}$ Player i plays a δ -good strategy. Let*

$$u_n^{-k} = \frac{1}{N-k} \sum_{j=k+1}^N u_n^j$$

be the average payoff to players $k+1, \dots, N$. Then

(i) For all $i \in \{1, \dots, k\}$:

$$0 \leq \liminf_{n \rightarrow \infty} u_n^{-k} - u_n^i \leq \limsup_{n \rightarrow \infty} u_n^{-k} - u_n^i \leq \frac{N-1}{N-k} \delta,$$

$$\limsup_{n \rightarrow \infty} u_n^i \leq v(C, N-1),$$

and, if mutual defection is inefficient,

$$\liminf_{n \rightarrow \infty} u_n^i \geq v(D, 0) - \delta.$$

(ii) Suppose $k = N$. Then:

(a)

$$L(\{u_n\}) \subset \text{diag}(E) = \{u \in E : u_1 = \dots = u_N\},$$

and if at least one of the players uses a continuous δ -good strategy, then

(b)

$$\lim_{n \rightarrow \infty} u_n = U(C, \dots, C) = (v(C, N-1), \dots, v(C, N-1)).$$

Let $\varepsilon > 0$. Let Θ^i be a strategy (as defined by equation (1)) for Player i .

The strategy profile $(\Theta^1, \dots, \Theta^N)$ is called an ε -Nash equilibrium if for every i and every alternative strategy Ξ^i for i , the payoff to i resulting from $(\Theta^1, \dots, \Theta^{i-1}, \Xi^i, \Theta^{i+1}, \dots, \Theta^N)$ cannot be ε better than the payoff resulting from $(\Theta^1, \dots, \Theta^N)$. More precisely:

For every $i \in \{1, \dots, N\}$, every strategy Ξ^i and every $\Sigma \times \Sigma$ -valued process $\{(s_n, \tilde{s}_n)\}$ adapted to the filtration $\{\mathcal{F}_n\}$ satisfying

$$P(s_{n+1} = s | \mathcal{F}_n) = \prod_{j=1}^N \Theta_n^j(s^j),$$

and

$$P(s_{n+1} = s | \mathcal{F}_n) = \left(\prod_{j \neq i} \Theta_n^j(s^j) \right) \Xi_n^i(s^i);$$

then

$$P(\limsup_{n \rightarrow \infty} \tilde{u}_n^i \leq \liminf_{n \rightarrow \infty} u_n^i + \varepsilon) = 1.$$

Here

$$u_n = \frac{1}{n} \sum_{k=1}^n U(s_k), \tilde{u}_n = \frac{1}{n} \sum_{k=1}^n U(\tilde{s}_k).$$

In other words, if all players but i play the equilibrium strategy, Player i cannot improve his payoff by more than ε if he deviates from Θ^i .

Corollary 3.2 *Let $\delta > 0$. Suppose that for all $i \in \{1, \dots, N\}$ Q^i is a continuous δ -good strategy. Then (Q^1, \dots, Q^N) is a $\delta(N-1)$ -Nash equilibrium.*

Proof: Follows from Theorem 3.1 □

Proof of Theorem 3.1

For all $\delta \geq 0$ let

$$\Lambda^i(\delta) = \{u \in E : -\delta \leq \mu^i(u) \leq 0\}.$$

Proposition 3.3 *Let $i \in \{1, \dots, N\}$. Suppose that i plays a δ -good strategy Q^i . Then $\Lambda^i(\delta)$ is a \mathcal{B} -set for Q^i . In particular, assertion (ii) of Theorem 2.2 and Theorem 2.4 hold.*

Proof: Suppose $i = 1$. Given $s^{-1} = (s^2, \dots, s^N) \in \{C, D\}^{N-1}$, let $k = \text{card}\{j > 1 : s^j = C\}$. Then

$$\mu^1(U(C, s^{-1})) = \frac{N - (k + 1)}{N - 1} (v(C, k) - v(D, k + 1)) \leq 0$$

and

$$\mu^1(U(D, s^{-1})) = \frac{k}{N - 1} (v(D, k) - v(C, k - 1)) \geq 0.$$

By Corollary 2.3 and definition of Q^1 , this concludes the proof. □

Proposition 3.4 (Properties of $\{\Lambda^i(\delta)\}$) *For all $\delta \geq 0$ the sets $\Lambda^i(\delta)$ satisfy the following properties:*

(i) For all $i \in \{1, \dots, N\}$ and $u \in \Lambda^i(\delta)$

$$u_i \leq v(C, N-1),$$

and, if mutual defection is Pareto inefficient,

$$u_i \geq v(D, 0) - \delta.$$

(ii) For all $k \leq N$, $u \in \bigcap_{i \in \{1, \dots, k\}} \Lambda^i(\delta)$ and $i \in \{1, \dots, k\}$

$$0 \leq \frac{\sum_{j=k+1}^N u_j}{N-k} - u_i \leq \delta \frac{N-1}{N-k}$$

(iii)

$$\bigcap_{i \in \{1, \dots, N\}} \Lambda^i(\delta) = \mathbf{diag}(E) = \{u \in E : u_1 = \dots = u_N\}.$$

Proof: Suppose $i = 1$. For all $\eta \in \mathbb{R}$ let Π^η be the orthogonal projection onto the hyperplan $\{\mu^1(u) = \eta\}$. Then

$$\Pi^\eta(u) = u - \left(\frac{\mu^1(u) - \eta}{\|\mu^1\|^2} \right) \mu^1$$

where μ^1 is the vector defined by $\mu^1(u) = \langle \mu^1, u \rangle$. That is $\mu^1 = (\mu_i^1)_{i=1, \dots, N}$ with $\mu_1^1 = 1$ and $\mu_i^1 = -\frac{1}{N-1}$ for $i > 1$. It follows that

$$\Pi_1^\eta(u) = u_1 - \frac{\mu^1(u) - \eta}{\|\mu^1\|^2} = \frac{\sum_{i=1}^N u_i}{N} + \eta \frac{N-1}{N}.$$

Thus, by Pareto dominance of $v(C, N-1)$

$$\Pi_1^\eta(u) \leq v(C, N-1) + \eta$$

for all $u = U(s)$, hence for all $u \in E$. Let now $u \in \Lambda^1$. Then $u = \Pi^\eta(u)$ with $\eta = \mu^1(u) \in [-\delta, 0]$. Thus $u_1 \leq v(C, N-1)$. Similarly, if $v(D, 0)$ is inefficient, then $u_1 \geq v(D, 0) - \delta \frac{N-1}{N}$.

To prove the second assertion set $A = \sum_{j=1}^k u_j$, $B = \sum_{j>k} u_j$ and note that, by definition of $\Lambda^i(\delta)$

$$-(N-1)\delta \leq Nu_i - A - B \leq 0$$

for all $i = 1, \dots, k$. Thus, by summing over all $i = 1, \dots, k$,

$$-(N-1)k\delta \leq (N-k)A - kB \leq 0.$$

Then

$$Nu_i \leq A + B \leq \frac{kB}{N-k} + B = \frac{NB}{N-k}$$

and

$$Nu_i \geq A + B - (N-1)\delta \geq \frac{k(B - \delta(N-1))}{N-k} + (B - (N-1)\delta) = \frac{N(B - (N-1)\delta)}{N-k}.$$

The last assertion is immediate, because on $\Lambda^i(\delta)$ $Nu_i \leq \sum_j u_j$. \square

Proof of Theorem 3.1 Assertion (i) and the beginning of (ii) follow from Propositions 3.3 and 3.4. It remains to prove the last assertion. Assume that Player 1 uses a continuous strategy. Recall that $C^1(x)$ is defined by (3) with $Q = Q^1$. By continuity of $x \mapsto Q_x^1$ the map $x \mapsto C^1(x)$ has a closed graph so that $\overline{\text{co}}(C)^1(x) = C^1(x)$ for all $x \in E$. Thus, by Theorem 2.1 (i), the set $L = L(\{u_n\})$ is invariant under the differential inclusion

$$\dot{u} \in -u + C^1(u) \tag{7}$$

and, by what precedes, is contained in $\text{diag}(E)$. In particular, for all $u \in L$ there exists η solution to (7) such that $\eta(0) = u$ and $\mu^1(\eta(t)) = 0$ for all t . Let $h(t) = \eta(t) + \dot{\eta}(t)$. Then $\mu^1(h(t)) = 0$ and $h(t) \in C^1(\eta(t))$ for almost all t . Let $v^* = U(C, \dots, C) = (v(C, N-1), \dots, v(C, N-1))$.

By definition of Q^1 and C^1 ,

$$\mu^1(u) \geq 0 \Rightarrow C^1(u) = \text{conv}\{U(C, s^{-1}) : s^{-1} \in \{C, D\}^{N-1}\}.$$

Now, the proof of Theorem 3.3 shows that $\mu^1(U(C, s^{-1})) \leq 0$ with equality only if $s = (C, \dots, C)$. Thus

$$\mu^1(u) \geq 0 \Rightarrow \{v \in C^1(u) : \mu^1(v) = 0\} = \{v^*\}.$$

This implies that $h(t) = v^*$ and $\eta(t) = e^{-t}(u - v^*) + v^*$ for all $t \in \mathbb{R}$. By compactness of L we must have $u = v^*$ (for otherwise $\{\eta(t)\}$ would be unbounded).

4 Network Prisoner's Dilemma Games

Network Games

In this section we consider a game in which players are located at the vertices of a graph and interact only with their neighbors. There are M players denoted $i = 1, \dots, M$. Player i has a finite action set Σ^i . The set $V = \{1, \dots, M\}$ of vertices of the graph is equipped with an *edge set* $\mathcal{E} \subset V \times V$.

We assume that the graph (V, \mathcal{E}) is

- (a) symmetric: $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$,
- (b) self-loop free: $(i, i) \notin \mathcal{E}$, and
- (c) irreducible: for all $i, j \in V$ there exist $k \geq 1$ and $i_1, \dots, i_k \in V$ such that $i_1 = i, i_k = j$ and $(i_l, i_{l+1}) \in \mathcal{E}$ for $l = 1, \dots, k-1$.

For each $(i, j) \in \mathcal{E}$ there is a real valued map

$$U^{ij} : \Sigma^i \times \Sigma^j \mapsto \mathbb{R}$$

representing the *payoff function to Player i against Player j* .

Let $\text{Neigh}(i) = \{j \in V : (i, j) \in \mathcal{E}\}$ and let N_i be its cardinal. The *payoff function to i* is the map $U^i : \Sigma \mapsto \mathbb{R}^{N_i}$ defined by

$$U^i(s) = (U^{ij}(s^i, s^j))_{j \in \text{Neigh}(i)}.$$

Using the notation of Example 1, set $N = \sum_{i=1}^M N_i$, and define the vector payoff function of the game as

$$U = (U^1, \dots, U^M) : \Sigma \mapsto \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_m} \simeq \mathbb{R}^N.$$

The state space of the game is then $E = \text{conv}\{U(s), s \in \Sigma\} \subset \mathbb{R}^N$.

In addition to these data, we assume given a Markov transition matrix $K = (K_{ij})_{i,j \in V}$ adapted to (V, \mathcal{E}) . That is

$$K_{ij} \geq 0, \quad \sum_j K_{ij} = 1$$

and

$$K_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}.$$

The *mean payoff* to Player i for the strategy profile s is defined as

$$\overline{U}^i(s) = \sum_j K_{ij} U^{ij}(s). \quad (8)$$

Irreducibility of the graph (V, \mathcal{E}) ensures irreducibility of the transition matrix K . Therefore there is a unique *invariant probability* π for K . That is,

$$\pi_i \geq 0, \sum_i \pi_i = 1$$

and for all $i \in V$

$$\sum_j \pi_j K_{ji} = \pi_i.$$

Define the *weight of edge* $(i, j) \in \mathcal{E}$ as

$$\omega_{ij} = \pi_i K_{ij}. \quad (9)$$

Such weights will prove to be useful for defining δ -good strategies below. Note that, by invariance of π ,

$$\sum_j \omega_{ij} = \sum_j \omega_{ji} = \pi_i \quad (10)$$

Example 3 Suppose

$$K_{ij} = \begin{cases} \frac{1}{N_i} & \text{if } j \in \text{Neigh}(i) \\ 0 & \text{if } j \notin \text{Neigh}(i) \end{cases}$$

Then

$$\overline{U}^i(s) = \frac{\sum_{j \in \text{Neigh}(i)} U^{ij}(s)}{N_i},$$

$$\pi_i = \frac{N_i}{N} \text{ and } \omega_{ij} = \frac{1}{N} \mathbf{1}_{j \in \text{Neigh}(i)}.$$

◇

Network Prisoner's Dilemma Games

We consider now a particular example of network games where each pair of neighboring players is engaged in two players prisoner dilemma game. We assume that for each $i \in V$ $\Sigma^i = \{C, D\}$, and

$$U^{ij}(C, D) = CD, \dots, U^{ij}(D, C) = DC,$$

where

(i)

$$CD < DD < CC < DC$$

as usual for the two player prisoner's dilemma game.

(ii) We furthermore assume that the outcome CC is *Pareto optimal* and that the outcome DD is *Pareto inefficient*, in the sense that for all $(i, j) \in \mathcal{E}$

$$(\omega_{ij} + \omega_{ji})DD < \omega_{ij}CD + \omega_{ji}DC < (\omega_{ij} + \omega_{ji})CC;$$

Remark 3 If K is reversible with respect to π (meaning that $\omega_{ij} = \omega_{ji}$) as in Example 3, Pareto inefficiency means

$$2DD < CD + DC < 2CC.$$

Equivalently, the polygon with vertices

$$(DD, DD), (CD, DC), (DC, CD), (CC, CC)$$

is convex and hence equal to E .

For $u = (u_{ij})_{i \in V, j \in \text{Neigh}(i)} \in \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_m}$

let

$$\mu^i(u) = \sum_j \omega_{ij}u_{ij} - \omega_{ji}u_{ji}.$$

Given $\delta \geq 0$, a δ -good strategy for Player i is a payoff-based strategy Q^i such that

$$Q_u^i(C) = 1 \text{ if } \mu^i(u) \geq 0,$$

and

$$Q_u^i(D) = 1 \text{ if } \mu^i(u) < -\delta.$$

The following result is similar to Theorem 3.1. It shows that if a group of players use δ -good strategies, their payoffs cannot be much worse than the payoff resulting from mutual defection and that a weighted average of the other players payoffs cannot be much better than hers. If furthermore, all the players play a δ -good strategy, and that of player i is continuous, then the payoffs of i against j and j against i both equal CC , given by mutual cooperation.

As a consequence (Corollary 4.2), continuous δ -good strategies form a Nash equilibrium. The proof is postponed to the end of the section.

Theorem 4.1 *Assume $1 \leq k \leq N$. Suppose that for $i \in \{1, \dots, k\}$, Player i plays a δ -good strategy. Then*

(i)

$$L(\{u_n\}) \subset \bigcap_{i \in \{1, \dots, k\}} \Lambda^i(\delta).$$

(ii)

$$DD - \frac{\delta}{2\pi_i} \leq \liminf_{n \rightarrow \infty} \bar{u}_n^i \leq \limsup_{n \rightarrow \infty} \bar{u}_n^i \leq CC, \quad (i = 1, \dots, k).$$

(iii)

$$\sum_{j=k+1}^N \pi_j DD \leq \liminf_{n \rightarrow \infty} \sum_{j=k+1}^N \pi_j \bar{u}_n^j \leq \limsup_{n \rightarrow \infty} \sum_{j=k+1}^N \pi_j \bar{u}_n^j \leq \sum_{j=k+1}^N \pi_j CC + \frac{k\delta}{2}.$$

(iv) *If $k = N$ and Player l uses a continuous δ -good strategy, then for all $j \in \text{Neigh}(l)$*

$$\lim_{n \rightarrow \infty} u_n^{lj} = \lim_{n \rightarrow \infty} u_n^{jl} = CC$$

Corollary 4.2 *Let $\delta > 0$. Suppose that for all $i \in \{1, \dots, N\}$ Q^i is a continuous δ -good strategy. Then (Q^1, \dots, Q^N) is a $\frac{(N-1)\delta}{2}$ Nash equilibrium.*

Proof of Theorem 4.1

For all $\delta \geq 0$ let

$$\Lambda^i(\delta) = \{u \in E : -\delta \leq \mu^i(u) \leq 0\}.$$

Proposition 4.3 *Let $i \in \{1, \dots, N\}$. Assume that i plays a δ -good strategy Q^i . Then $\Lambda^i(\delta)$ is a \mathcal{B} -set for Q^i .*

Proof: Fix $i \in V$. Let $s = (s^1, \dots, s^M) \in \Sigma^1 \times \dots \times \Sigma^M$ be such that $s^i = C$. Then

$$\mu^i(U(s)) = \sum_j \omega_{ij}(CCt_j + CD(1 - t_j)) - \sum_j \omega_{ji}(CCt_j + DC(1 - t_j))$$

where $t_j = 1$ if $s^j = C$ and 0 otherwise. Thus

$$\begin{aligned} \mu^i(U(s)) &= \sum_j \omega_{ij}t_j(CC - CD) + \sum_j \omega_{ji}t_j(DC - CC) + CD \sum_j \omega_{ij} - DC \sum_j \omega_{ji} \\ &\leq \sum_j \omega_{ij}(CC - CD) + \sum_j \omega_{ji}(DC - CC) + CD \sum_j \omega_{ij} - DC \sum_j \omega_{ji} \\ &= \pi_i(CC - CD + DC - CC + CD - DC) = 0. \end{aligned}$$

Suppose now that $s^i = D$. Then

$$\begin{aligned} \mu^i(U(s)) &= \sum_j \omega_{ij}(DCt_j + DD(1 - t_j)) - \sum_j \omega_{ji}(CDt_j + DD(1 - t_j)) \\ &= \sum_j \omega_{ij}t_j(DC - DD) + \sum_j \omega_{ji}t_j(DD - CD) \geq 0. \end{aligned}$$

The results then follows from Corollary 2.3. \square

Remark 4 the proof above shows that $\mu^i(U(s)) < 0$ (respectively > 0) if $s^i = C$ (resp. D) and $s^j = D$ (resp. C) for some $j \neq i$

Proposition 4.4 (Properties of $\{\Lambda^i(\delta)\}$) *For all $\delta \geq 0$ the sets $\{\Lambda^i(\delta)\}$ verify the following properties:*

(i) *For $i \in \{1, \dots, N\}$ and $u \in \Lambda^i(\delta)$ set $\bar{u}_i = \sum_j K_{ij}u_{ij}$. Then*

$$DD - \frac{\delta}{2\pi_i} \leq \bar{u}_i \leq CC.$$

(ii) For all $k < N$ and $u \in \bigcap_{j=1}^k \Lambda^j(\delta)$

$$0 \leq \sum_{i=k+1}^N \mu^i(u) \leq k\delta,$$

$$\left(\sum_{i=k+1}^N \pi_i\right)DD \leq \sum_{i=k+1}^N \pi_i \bar{u}_i \leq \left(\sum_{i=k+1}^N \pi_i\right)CC + \frac{k\delta}{2}$$

(iii)

$$\bigcap_{i \in \{1, \dots, N\}} \Lambda^i(\delta) = \bigcap_{i \in \{1, \dots, N\}} \Lambda^i(0)$$

Proof: (i). Let $v \in \mathbb{R}^N$ be the vector defined by $v_{ij} = 1, v_{ji} = -1$ for all $j \in \text{Neigh}(i)$ and $v_{kl} = 0$ if $k \neq i, l \neq i$ or $(k, l) \notin \mathcal{E}$. Let Π^η be the projection onto the hyperplan $\{\mu^i(u) = \eta\}$ parallel to v . That is

$$\Pi^\eta(u) = u - \frac{\mu^i(u) - \eta}{\mu^i(v)} v = u - \frac{\mu^i(u) - \eta}{2\pi_i} v.$$

Thus, for all $s \in \Sigma$

$$\sum_j \omega_{ij} \Pi^\eta(U(s))_{ij} = \frac{\sum_j (\omega_{ij} U^{ij}(s) + \omega_{ji} U^{ji}(s)) + \eta}{2}$$

$$\leq \frac{\sum_j (\omega_{ij} + \omega_{ji}) CC + \eta}{2} = \pi_i CC + \frac{\eta}{2}, \quad (11)$$

$$\geq \frac{\sum_j (\omega_{ij} + \omega_{ji}) DD + \eta}{2} = \pi_i DD + \frac{\eta}{2} \quad (12)$$

where the last inequalities follow from Pareto dominance. This implies that for all $u \in E$

$$DD \frac{\eta}{2\pi_i} \leq \sum_j K_{ij} \Pi^\eta(u)_{ij} = \frac{1}{\pi_i} \sum_j \omega_{ij} \Pi^\eta(u)_{ij} \leq CC + \frac{\eta}{2\pi_i}.$$

Hence for all $u \in \Lambda^i$

$$DD - \frac{\delta}{2\pi_i} \leq \sum_j K_{ij} u_{ij} \leq CC.$$

(ii). Note that

$$\sum_j \mu^j(u) = \sum_j \sum_k \omega_{jk} u_{jk} - \omega_{kj} u_{kj} = \sum_k \sum_j \omega_{jk} u_{jk} - \omega_{kj} u_{kj} = - \sum_j \mu^j(u).$$

Thus, $\sum_j \mu^j(u) = 0$ and the inequalities follow from the definition of $\Lambda^j(\delta)$ for the first one and inequalities (11, 12) for the second one.

(iii). Let $u \in \bigcap \Lambda^j(\delta)$. Then $\mu^i(u) \leq 0$ for all i , but since $\sum_i \mu^i(u) = 0$, $\mu^i(u) = 0$. \square

Proof of Theorem 4.1 The proof is similar to the proof of Theorem 3.1. Assertion (i), (ii) and (iii) follow from Propositions 4.3 and 4.4. For (iv) we use the fact that if player 1 plays a continuous δ good strategy, then the limit set L of $\{u_n\}$ is an invariant set of the differential inclusion $\dot{u} \in -u + C^1(u)$ contained in $\bigcap \Lambda^i(0)$. By proposition 4.3 and remark 4, for all $u \in \bigcap_i \Lambda^i(0)$ and $v \in C^1(u)$ $u^{1j} = u^{j1} = CC$. Thus, reasoning like in the proof of Theorem 3.1, invariance of L shows that for all $u \in L$ $u^{1j} = u^{j1} = CC$.

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